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Core in a Cooperative Dynkin's Stopping Problem

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Abstract. We consider multiperson cooperative stopping game of Dynkin's type. We are interested in Pareto optimal stopping times which dominates a conservative value for each player. The set of such a Pareto optimal stopping times is called core. Such a core is necessarily nonempty. We first give necessary and sufficient conditions for the core to be nonempty. Secondly we give a characterization of core. Also, by the method of scalarization we find ε-Pareto optimal stopping times for each player.

1. Introduction.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_n)_{n\in N}$ an increasing family of sub-σ-fields of $\mathcal{F}$, where $N = \{0, 1, 2, \ldots\}$ is a discrete time space.

Let $X(\mathbf{n}) = ((X_1(\mathbf{n}), X_2(\mathbf{n}), ..., X_p(\mathbf{n})) : \mathbf{n} \in N^p)$ be a vector-valued stochastic process on $(\Omega, \mathcal{F}, P)$ and on p-dimensional discrete time space N^p such that $X(\mathbf{n})$ is $\mathcal{F}_{\min_i n_i}$-measurable and $\sup_{\mathbf{n}\in N^p} \max_i \mid X_i(\mathbf{n}) \mid$ is integrable, where $\mathbf{n} = (n_1, n_2, ..., n_p) \in N^p$.

For each $k \in N$, we denote by Λ_k the class of $(\tau_1, \tau_2, ..., \tau_p)$ such that τ_i $(i = 1, 2, ..., p)$ is $(\mathcal{F}_n)$-stopping time and $k \leq \min_i \tau_i < \infty$ a.s..

Now we consider the following coopetative stopping game. There are p players and each player i chooses stopping time τ_i $(i = 1, 2, ..., p)$ such that $(\tau_1, \tau_2, ..., \tau_p) \in \Lambda_0$. Then the ith player $(i = 1, 2, ..., p)$ gets the reward $X_i(\tau_1, \tau_2, ..., \tau_p)$. The aim of the ith player is to maximize the expected gain $E[X_i(\tau_1, \tau_2, ..., \tau_p)]$ with respect to τ_i, cooperating with other players. However, the stopping time chosen by one of them generally depends upon one decided by other, even if they cooperate. Thus we shall use the concept of Pareto optimality as in the usual cooperative game of the game thery or the multiobjective problem of mathematical programming.

2. Core.

Before giving the definition of Pareto optimality, we define partial orders in the $p-$ dimensional Euclidean space as follows: for two vectors $x = (x_1, x_2, ..., x_p)$ and $y = (y_1, y_2, ..., y_p)$, $x > y$ if $x_i > y_i$, for all i ; $x \geqq y$ if $x_i \geqq y_i$, for all i ; $x = y$ if $x_i = y_i$, for all i ; $x \geq y$ if $x \geqq y$ and $x \neq y$.

We define a conditional expected reward by $G_n^i(\tau_1, \tau_2, ..., \tau_p) = E[X_i(\tau_1, \tau_2, ..., \tau_p) \mid \mathcal{F}_n]$ for player i $(i = 1, 2, ..., p)$, and a vector by

$$G_n^*(\tau_1, \tau_2, ..., \tau_p) = (G_n^1(\tau_1, \tau_2, ..., \tau_p), G_n^2(\tau_1, \tau_2, ..., \tau_p), ..., G_n^p(\tau_1, \tau_2, ..., \tau_p))$$

and let $e = (1, 1, ..., 1)$.

For the sake of simplicity, without further comments we assume that all inequalities and equalities between random variables hold in the sense of "almost surely".

For $n \in N$ and $\varepsilon \geqq 0$, we say that $(\tau_1^\varepsilon, \tau_2^\varepsilon, ..., \tau_p^\varepsilon)$ in Λ_n is ε*–weak* (resp. *strong*) *Pareto optimal at* n if there is no $(\tau_1, \tau_2, ..., \tau_p)$ in Λ_n such that

$$G_n^*(\tau_1, \tau_2, ..., \tau_p) > G_n^*(\tau_1^\varepsilon, \tau_2^\varepsilon, ..., \tau_p^\varepsilon) + \varepsilon e,$$

$$(resp. G_n^*(\tau_1, \tau_2, ..., \tau_p) \geq G_n^*(\tau_1^\varepsilon, \tau_2^\varepsilon, ..., \tau_p^\varepsilon) + \varepsilon e).$$

We shall simply call a 0–weak (resp. 0–strong) Pareto optimal pair a weak (resp. strong) Pareto optimal one.

Next, we introduce a core which is a subset of Pareto optimal pairs. Let $Z = (Z_n : n \in N) = ((Z_n^1, Z_n^2, ..., Z_n^p) : n \in N)$ be a vector-valued stochastic process on $(\Omega, \mathcal{F}, P)$ such that $(Z_n^i : n \in N)$ is adapted to $(\mathcal{F}_n)$ and bounded. For $\varepsilon \geqq 0$, we define ε*–core* $C_n^\varepsilon(Z)$ at time n by the class of all $(\tau_1^\varepsilon, \tau_2^\varepsilon, ..., \tau_p^\varepsilon)$ in Λ_n such that $(\tau_1^\varepsilon, \tau_2^\varepsilon, ..., \tau_p^\varepsilon)$ is ε–weak Pareto optimal at n and inequality

$$G_n^*(\tau_1^\varepsilon, \tau_2^\varepsilon, ..., \tau_p^\varepsilon) \geq Z_n - \varepsilon e \tag{1}$$

holds. This Z^i is one called threat functional and is interpreted as a minimum value which the ith player is able to compromise with.

In general ε–core $C_n^\varepsilon(Z)$ may be empty, even if ε is positive and $Z_n^i \leqq \alpha_n^i$ $(i = 1, 2, ..., p)$. For example, when $p = 2$, $X_1(n, k) = a, k \geq n, X_1(k, n) = b, k \geq n+1, X_2(k, n) = a, k \geq n, X_2(n, k) = b, k \geq n+1$, and $Z_n^i = c$ $(i = 1, 2, n \in N)$ for constants a, b and c satisfying $a < c < b$, we have $\mathcal{G}_n = \{(a, b), (b, a), (a, a)\}$, $n \in N$, and vectors (a, b) and (b, a) correspond weak (and strong) Pareto optimal pairs. Here $\mathcal{G}_n = \{G_n^*(\tau_1, \tau_2, ..., \tau_p) \mid (\tau_1, \tau_2, ..., \tau_p) \in \Lambda_n\}$. However, since there is no pair $(\tau_\varepsilon, \sigma_\varepsilon)$ satisfying (1) for sufficiently small ε, ε–core $C_n^\varepsilon(Z)$ is empty.

In this section we give necessary and sufficient conditions for ε–core $C_n^\varepsilon(Z)$ $(\varepsilon > 0)$ to be nonempty, and find $(\tau_1^*, \tau_2^*, ..., \tau_p^*)$ in $C_n^0(Z)$.

To end this, for given other bounded processes $M^i = (M_n^i)$, $i = 1, 2, ..., p$ and $(\tau_1, \tau_2, ..., \tau_p) \in \Lambda_n$ we define random variables by, if these exist,

$$\gamma_n^i(\tau_1, \tau_2, ..., \tau_p) = \frac{M_n^i - G_n^*(\tau_1, \tau_2, ..., \tau_p)}{M_n^i - Z_n^i}, \quad i = 1, 2, ..., p$$

$$\gamma_n(\tau_1, \tau_2, ..., \tau_p) = \max_i \{\gamma_n^i(\tau_1, \tau_2, ..., \tau_p)\},$$

and a minimum value process $\gamma^* = (\gamma_n^*)$ by

$$\gamma_n^* \equiv \gamma_n^*(M, Z) = \operatorname*{ess\,inf}_{(\tau_1, \tau_2, ..., \tau_p) \in \Lambda_n} \gamma_n(\tau_1, \tau_2, ..., \tau_p).$$

Here M^i may be a goal. The following assumption is natural in our problem.

ASSUMPTION 2.1. $M_n^i \geqq \alpha_n^i \geqq Z_n^i$ and $M_n^i > Z_n^i$ for all i an all $i = 1, 2, ..., p$, where $\alpha_n^i = \text{ess sup}_{(\tau_1, \tau_2, ..., \tau_p) \in \Lambda_n} G_n^i(\tau_1, \tau_2, ..., \tau_p)$.

If Assumption 2.1 is satisfied, γ_n^* is nonnegative, but it is not necessarily less than or equal to 1. Indeed, in the above example, letting $M_n^i = b,\ i = 1, 2,\ n \in N$, we have $\gamma_n^* = (b-a)/(b-c) > 1,\ n \in N$.

ASSUMPTION 2.2. Processes $M^i - Z^i$ $(i = 1, 2, ..., p)$ are bounded from above, that is, there is a constant L such that $M_n^i - Z_n^i \leqq L$ for all i and all $n \in N$.

THEOREM 2.1. *Suppose Assumptions* 2.1 *and* 2.2 *are satisfied. For each* $n \in N$, *the following conditions are equivalent :*

(a) *For each* $\varepsilon > 0$, ε*–core* $C_n^\varepsilon(Z)$ *is nonempty.*

(b) *For each* $\varepsilon > 0$, *there exists a* $(\tau_1^\varepsilon, \tau_2^\varepsilon, ..., \tau_p^\varepsilon)$ *in* Λ_n *satisfying inequality* (1).

(c) $\gamma_n^* \leqq 1$.

Furthermore, if one of conditions (a),(b) *and* (c) *is satisfied, a* $(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon)$ *in* Λ_n *such that* $\gamma_n^* \geqq \gamma_n(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon) - \varepsilon/L$ *is an element of* $C_n^\varepsilon(Z)$ *for each* $n \in N$*and every* $\varepsilon > 0$.

PROOF. By the definition of ε–core $C_n^\varepsilon(Z)$, the implication (a) $\Rightarrow$ (b) is immediate. (b) $\Rightarrow$ (c). From inequality (1), we have $\gamma_n^i(\tau_1^\varepsilon, \tau_2^\varepsilon, ..., \tau_p^\varepsilon) \leqq 1 + \varepsilon(M_n^i - Z_n^i)^{-1}$ for every i, so that

$$\gamma_n^* \leqq \gamma_n(\tau_1^\varepsilon, \tau_2^\varepsilon, ..., \tau_p^\varepsilon) \leqq 1 + \varepsilon \max_i (M_n^i - Z_n^i)^{-1}.$$

Letting as $\varepsilon \downarrow 0$, we have the desired inequality $\gamma_n^* \leqq 1$.
(c) $\Rightarrow$ (a). By the definition of γ_n^*, there is a $(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon)$ in Λ_n such that

$$\gamma_n^* \geqq \gamma_n(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon) - \varepsilon/L.$$

Thus since $\gamma_n^* \leqq 1$, we have

$$G_n^i(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon) \geqq Z_n^i - \varepsilon(M_n^i - Z_n^i)/L \geqq Z_n^i - \varepsilon, \quad i = 1, 2, ..., p,$$

which implies that $(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon)$ satisfies inequality (1). Next we assume that this pair $(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon)$ is not ε–weak Pareto optimal at n, that is, there exists a $(\tau_1, \tau_2, ..., \tau_p)$ in Λ_n satisfying $G_n^i(\tau_1, \tau_2, ..., \tau_p) > G_n^i(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon) + \varepsilon$ for every i. Then we have

$$\gamma_n^i(\tau_1, \tau_2, ..., \tau_p) < \gamma_n^i(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon) - \varepsilon/L \leqq \gamma_n(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon) - \varepsilon/L, \quad i = 1, 2, ..., p,$$

so that

$$\gamma_n(\tau_1, \tau_2, ..., \tau_p) < \gamma_n(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon) - \varepsilon/L \leqq \gamma_n^*,$$

which is contrary to the fact that in general $\gamma_n(\tau_1, \tau_2, ..., \tau_p) \geqq \gamma_n^*$. Hence $(\hat{\tau}_1^\varepsilon, \hat{\tau}_2^\varepsilon, ..., \hat{\tau}_p^\varepsilon)$ is ε–weak Pareto optimal at n, and it is in $C_n^\varepsilon(Z)$. Therefore ε–core $C_n^\varepsilon(Z)$ is nonempty.

The proof of the second statement is given in that of the implication (c) $\Rightarrow$ (a). □

In the following theorem, we give a characterization of an element in $C_n^0(Z)$.

THEOREM 2.2. *Suppose that Assumption* 2.1 *is satisfied and that* $\gamma_n^* \leqq 1$ *for every* $n \in N$. *For each* $n \in N$, *a* $(\tau_1^*, \tau_2^*, ..., \tau_p^*)$ *in* Λ_n *satisfies* $\gamma_n^* = \gamma_n(\tau_1^*, \tau_2^*, ..., \tau_p^*)$ *if and only if*

$$G_n^i(\tau_1^*, \tau_2^*, ..., \tau_p^*) \geqq (1 - \gamma_n^*)M_n^i + \gamma_n^* Z_n^i, \quad i = 1, 2, ..., p, \tag{2}$$

where the equality holds for at least one i. *Furthermore, such a* $(\tau_1^*, \tau_2^*, ..., \tau_p^*)$ *is in* $C_n^0(Z)$.

PROOF. If $\gamma_n^* = \gamma_n(\tau_1^*, \tau_2^*, ..., \tau_p^*)$, we have

$$\gamma_n^* \geqq \gamma_n^i(\tau_1^*, \tau_2^*, ..., \tau_p^*), \quad i = 1, 2, ..., p,$$

where at least one i have equality (as well as in the inequality below), and hence

$$G_n^i(\tau_1^*, \tau_2^*, ..., \tau_p^*) \geqq M_n^i - \gamma_n^*(M_n^i - Z_n^i) = (1 - \gamma_n^*)M_n^i + \gamma_n^* Z_n^i, \quad i = 1, 2, ..., p.$$

Conservely, if a $(\tau_1^*, \tau_2^*, ..., \tau_p^*)$ satisfies (2), it is clear that $\gamma_n^* = \gamma_n(\tau_1^*, \tau_2^*, ..., \tau_p^*)$. Next, by argument analogous to the proof of Theorem 2.1 it is easy to see without Assumption 2.2 that the $(\tau_1^*, \tau_2^*, ..., \tau_p^*)$ is in $C_n^0(Z)$. □

3. fundamental lemmas.

In this section we give fundamental results, in order to obtain properties of shadow (virtural) optimum and to use these results in the last section. We first define shadow optimum α^i for the reward $X_i(\tau_1, \tau_2, \ldots, \tau_p)$ as follows:

$$\alpha_n^i = \operatorname*{ess\,sup}_{(\tau_1, \tau_2, ..., \tau_p) \in \Lambda_n} G_n^i(\tau_1, \tau_2, ..., \tau_p), \quad n \in N, \ i = 1, 2, \ldots, p.$$

In multiobjective programming, the shadow optima are also called "ideal solution".

Now, to obtain constructive property of the shadow optima, we generally consider an optimal stopping problem so as to maximize the expected reward $G_n(\tau_1, \tau_2, ..., \tau_p) = E[X(\tau_1, \tau_2, ..., \tau_p)|\mathcal{F}_n]$ with respect to $(\tau_1, \tau_2, ..., \tau_p) \in \Lambda_n$, where $X(n_1, \ldots, n_p)$ satisfies the same conditions as $X_i(n_1, \ldots, n_p)$. The optimal value process $\beta = (\beta_n)_{n \in N}$ is defined by

$$\beta_n = \operatorname*{ess\,sup}_{(\tau_1, \tau_2, ..., \tau_p) \in \Lambda_n} G_n(\tau_1, \tau_2, ..., \tau_p), \quad n \in N.$$

For $n \in N$ and $\varepsilon \geqq 0$, we say that a pair $(\tau_1^\varepsilon, \tau_2^\varepsilon, \ldots, \tau_p^\varepsilon)$ in Λ_n is (ε, β)*–optimal at* n if $\beta_n \leqq G_n(\tau_1^\varepsilon, \tau_2^\varepsilon, \ldots, \tau_p^\varepsilon) + \varepsilon$.

ASSUMPTION 3.1. For each $n \in N$ and every $(n_1, n_2, \ldots, n_p) \in N^p$ such that $\min_i n_i = n$,

$$X(n_1, \ldots, n_p) \leqq \tilde{X}_n,$$

where

$$\tilde{X}_n = \max_{n_j = n, n+1; \min_k n_k = n} X(n_1, \ldots, n_p).$$

LEMMA 3.1. *Suppose Assumptions* 3.1 *is satisfied.*

(i) *The process* $\beta = (\beta_n)$ *satisfies the recursive relation:*

$$\beta_n = \max(\tilde{X}_n, E[\beta_{n+1} \mid \mathcal{F}_n]), \quad n \in N. \tag{3}$$

(ii) β *is the smallest supermartingale dominating the process satisfying* (3).

(iii) $\limsup_n \beta_n = \liminf_n \tilde{X}_n$.

PROOF. The lemma is easily proved as in the classical optimal stopping problem (cf. Chow, Robbins and Siegmund [2] or Neveu [8]). □

From this lemma it is easy to see that the process β coincides with an optimal value process $\hat{\beta} = (\hat{\beta}_n)$ in an optimal stopping problem with a reward $\tilde{X}_n$ of time n, i. e.

$$\hat{\beta}_n = \operatorname*{ess\,sup}_{n \leq \tau < \infty} E[\tilde{X}_\tau \mid \mathcal{F}_n].$$

Hence $\beta = \hat{\beta}$ is constructive by the method of the backward induction as in Chow and et. al. [2].

For each $n \in N$and $\varepsilon \geqq 0$, define stopping times $\tau_i^\varepsilon(n) \equiv \tau_i^\varepsilon(n, \beta)$ by

$$\tau_i^\varepsilon(n) = \inf\{k \geqq n \mid \beta_k \leqq \max_{n_j = k, k+1; j \neq i} X(n_1, \ldots, n_{i-1}, k, n_{i+1}, \ldots, n_p) + \varepsilon\}$$

where $\inf(\phi) = +\infty$.

LEMMA 3.2. *Suppose Assumptions* 3.1 *is satisfied and let* $n \in N$*be arbitrary.*

(i) *For each* $\varepsilon > 0$*, the pair* $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n))$ *is* (ε, β)*–optimal at* n*.*

(ii) *The stopping time* $\min_i \tau_i^0(n)$ *is a. s. finite, the pair* $(\tau_1^0(n), \tau_2^0(n), \ldots, \tau_p^0(n))$ *is* $(0,\beta)$–*optimal at* n.

PROOF. When ε is positive, it follows from Lemma 3.1 (iii) that the stopping time $\min_i \tau_i^\varepsilon(n)$ is a. s. finite. Thus, for $\varepsilon \geqq 0$, it suffices to show that inequality $\beta_n \leqq G_n(\tau_1^\varepsilon, \tau_2^\varepsilon, \ldots, \tau_p^\varepsilon) + \varepsilon$ holds for each $n \in N$. From Lemma 3.1 (i) and the optional sampling theorem, we have $\beta_n = E[\beta_{\tau_1^\varepsilon(n)\wedge\tau_2^\varepsilon(n)\wedge\ldots\wedge\tau_p^\varepsilon(n))} \mid \mathcal{F}_n]$. Furthermore, since $\beta_k \leqq X(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_i^\varepsilon(n), \ldots, \tau_p^\varepsilon(n)) + \varepsilon$ on $\{\tau_i^\varepsilon(n) = k\}$, we have the desired inequality. □

4. Scalarization and Pareto optima.

In this section we find Pareto optimal pairs by the method of the well–known scalarization.

Let S denote the set of vectors $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ in $\mathbf{R}^p$ satisfying $\lambda \geq 0$ and $\sum_i \lambda_i = 1$, and S_0 the set of λ in S such that $\lambda > 0$. For given $X_i(\mathbf{n}), \mathbf{n} \in N^p, i = 1, 2, \ldots, p$, and λ in S, we define sequences of random variables by

$$X(\mathbf{n}; \lambda) = \sum_{i=1}^{p} \lambda_i X_i(\mathbf{n}),$$

and for $(\tau_1, \tau_2, \ldots, \tau_p) \in \Lambda_n$, let

$$G_n(\tau_1, \tau_2, \ldots, \tau_p; \lambda) = \sum_{i=1}^{p} \lambda_i G_n^i(\tau_1, \tau_2, \ldots, \tau_p) = E[X(\tau_1, \tau_2, \ldots, \tau_p; \lambda) | \mathcal{F}_n].$$

Then a maximum value process is defined by

$$V_n(\lambda) = \operatorname*{ess\,sup}_{(\tau_1, \tau_2, \ldots, \tau_p) \in \Lambda_n} G_n(\tau_1, \tau_2, \ldots, \tau_p; \lambda), \quad n \in N.$$

We also define stopping times for the process $V(\lambda) = (V_n(\lambda))$ as follows:

$$\tau_i^\varepsilon(n) = \inf\{k \geqq n \mid V_k(\lambda) \leqq \max_{n_j = k, k+1; j \neq i} X(n_1, \ldots, n_{i-1}, k, n_{i+1}, \ldots, n_p; \lambda) + \varepsilon\}$$

for $n \in N$ and $\varepsilon \geqq 0$. The following theorems are immediate results of Lemmas 3.1 and 3.2.

ASSUMPTION 4.1. For each $n \in N$ and every $(n_1, n_2, \ldots, n_p) \in N^p$ such that $\min_i n_i = n$ and all $i = 1, 2, \ldots, p$,

$$X_i(n_1, \ldots, n_p) \leqq \tilde{X}_n^i,$$

where

$$\tilde{X}_n^i = \max_{n_j = n, n+1; \min_k n_k = n} X_i(n_1, \ldots, n_p).$$

Let

$$\tilde{X}_n(\lambda) = \max_{n_j = n, n+1; \min_k n_k = n} X(n_1, \ldots, n_p; \lambda).$$

Then we easily see that if Assumption 4.1 is satisfied, the relation

$$X(n_1, \ldots, n_p; \lambda) \leqq \tilde{X}_n(\lambda)$$

holds.

The following theorems are immediate results of Lemmas 3.1 and 3.2.

THEOREM 4.1. *Suppose Assumptions* 4.1 *is satisfied let* λ *in* S *be arbitrary.*

(i) *The process* $V(\lambda) = (V_n(\lambda))$ *satisfies the recursive relation:*

$$V_n(\lambda) = \max(\tilde{X}_n(\lambda), E[V_{n+1}(\lambda) \mid \mathcal{F}_n]), \quad n \in N. \tag{4}$$

(ii) $V(\lambda)$ *is the smallest supermartingale satisfying* (4).

(iii) $\limsup_n V_n(\lambda) = \liminf_n \tilde{X}_n(\lambda)$.

THEOREM 4.2. *Suppose Assumptions* 4.1 *is satisfied, let* $n \in N$ *and* $\lambda \in S$ *be arbitrary.*

(i) *For each* $\varepsilon > 0$*, the pair* $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n))$ *is* $(\varepsilon, V(\lambda))$*-optimal at* n*.*

(ii) *The stopping time* $\min_i \tau_i^0(n)$ *is a. s. finite, the pair* $(\tau_1^0(n), \tau_2^0(n), \ldots, \tau_p^0(n))$ *is* $(0, V(\lambda))$*-optimal at* n*.*

The general lemma below is a well-known result in multiobjective problem.

LEMMA 4.1. *Let* $n \in N$*,* $\varepsilon \geqq 0$ *and* $\lambda \in S$ *be arbitrary. If a pair* $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n))$ *in* Λ_n *satisfies inequality* $V_n(\lambda) \leqq G_n(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n); \lambda) + \varepsilon$*, then the pair* $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n))$ *is* ε*–weak Pareto optimal at* n*. Furthermore when* λ *is in* S_0*, the pair* $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n))$ *is* ε*–strong Pareto optimal at* n*.*

PROOF. We suppose that the pair $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n))$ is not ε–weak Pareto optimal. There then exists a pair $(\tau_1, \tau_2, \ldots, \tau_p)$ in Λ_n such that $G_n^*(\tau_1, \tau_2, \ldots, \tau_p) > G_n^*(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n)) + \varepsilon e$, that is, $G_n^i(\tau_1, \tau_2, \ldots, \tau_p) > G_n^i(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n)) + \varepsilon$ for every $i = 1, 2, \ldots, p$. Thus we have

$$\begin{aligned} G_n(\tau_1, \tau_2, \ldots, \tau_p; \lambda) &= \sum_{i=1}^{p} \lambda_i G_n^i(\tau_1, \tau_2, \ldots, \tau_p) \\ &> \sum_{i=1}^{p} \lambda_i G_n^i(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n)) + \varepsilon \\ &= G_n(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n); \lambda) + \varepsilon, \end{aligned}$$

so that $V_n(\lambda) > G_n(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n); \lambda) + \varepsilon$, which is a contradiction. Hence the pair $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n))$ is ε–weak Pareto optimal. Similarly, the statement for $\lambda > 0$ is proved. □

Theorem 4.2 and Lemma 4.1 immediately imply the following theorem.

THEOREM 4.3. *Suppose Assumptions* 4.1 *is satisfied, let* $n \in N$ *and* $\lambda \in S$ *be arbitrary.*

(i) *For each* $\varepsilon > 0$, *the pair* $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n))$ *is* ε*–weak Pareto optimal at* n ; *if in addition* λ *is in* S_0 *then the pair* $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \ldots, \tau_p^\varepsilon(n))$ *is* ε*–strong Pareto optimal at* n.

(ii) *If the stopping time* $\min_i \tau_i^0(n)$ *is a. s. finite, the pair* $(\tau_1^0(n), \tau_2^0(n), \ldots, \tau_p^0(n))$ *is weak Pareto optimal at* n ; *if in addition* λ *is in* S_0 *then the pair* $(\tau_1^0(n), \tau_2^0(n), \ldots, \tau_p^0(n))$ *is strong Pareto optimal at* n.

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